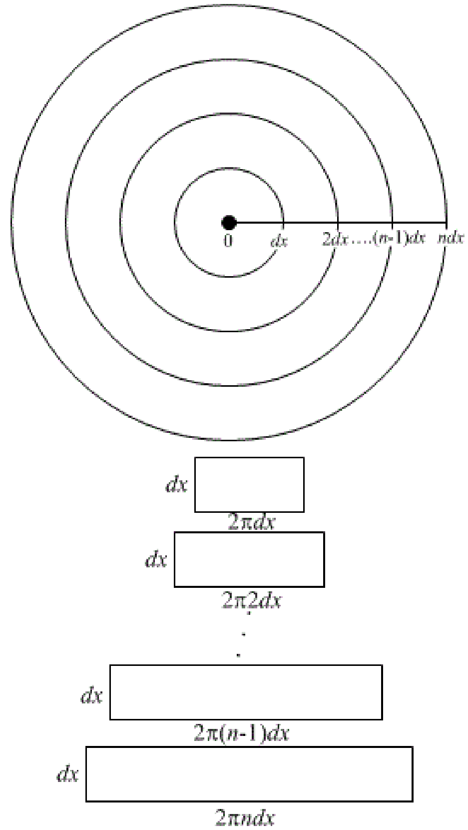
Deriving π Using Area of a Circle



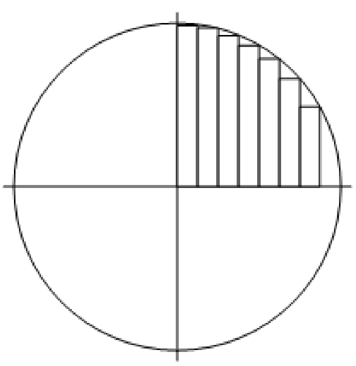
As a student in the 1980's I wanted to prove that the area of a circle is indeed a constant times the radius squared, but what I ended up doing was deriving that constant. I started off with the assumption that the circumference of a circle is proportional to its radius. This seems reasonable since the perimeter of an *n*-sided regular polygon is proportional to its diameter and a circle is the limit as *n* goes towards infinity of an *n*-sided regular polygon. Let's go ahead and call that constant 2π.

Break up the circle into *n* cocentric circles with adjacent circles having a difference of radius of *dx*. Since we have *n* of these circles, we know $dx=r/n$. Now we calculate the area of circles by cutting along the circular lines and unrolling each piece. As *n* goes to infinity the top and bottom of each unrolled piece will get closer in size to each other thus making the unrolled pieces approximate a rectangle. Thus

$$A \approx \sum_{i=1}^{n}(2\pi i\,dx)(dx)=\sum_{i=1}^{n}(2\pi i)(dx)^2=2\pi\sum_{i=1}^{n} i\left(\tfrac{r}{n}\right)^2=\frac{2\pi r^2}{n^2}\sum_{i=1}^{n} i$$

$$=\frac{2\pi r^2}{n^2}\cdot\frac{1+n}{2}\cdot n=\frac{\pi(1+n)r^2}{n}$$

Next, take the limit as *n* goes to infinity of the above expression to get $A = \pi r^2$. So the original objective of showing that the area of a circle is proportional to the square of its radius has been achieved. That means that the area of a circle is a constant times its radius squared and we can call that constant π. The next object is to derive π. We can accomplish this by using another method for calculating the area of the circle and then setting the two methods equal to each other.



We could also calculate the area of a circle by taking the limit of the Reimann Sum of the rectangles in the first quadrant and multiplying that area by 4. If we center the circle at the origin, the equation of the circle is: $x^2 + y^2 = r^2$. This leads to an area equation of:

$$A \approx 4\sum_{i=1}^{n}(dx)\sqrt{r^2-x^2}=4\sum_{i=1}^{n}\sqrt{r^2-(idx)^2}\,dx=4\sum_{i=1}^{n}\sqrt{r^2-\left(\tfrac{ir}{n}\right)^2}\,\frac{r}{n}=4\sum_{i=1}^{n}\sqrt{\frac{r^2n^2-i^2r^2}{n^2}}\,\frac{r}{n}$$

$$=4\sum_{i=1}^{n}\sqrt{n^2-i^2}\,\frac{r^2}{n^2}=\frac{4r^2}{n^2}\sum_{i=1}^{n}\sqrt{n^2-i^2}\quad so\ A=\lim_{n\to\infty}\frac{4r^2}{n^2}\sum_{i=1}^{n}\sqrt{n^2-i^2}$$

$$=4r^2\lim_{n\to\infty}\frac{1}{n^2}\sum_{i=1}^{n}\sqrt{n^2-i^2}$$

Setting this equals to πr^2 gives us: $\pi = 4\lim_{n\to\infty}\frac{1}{n^2}\sum_{i=1}^{n}\sqrt{n^2-i^2}$. When I first came up with this formula, I owned a TI-58C calculator.

I program it with the formula and left it plugged in for several days after putting a value in for *n* large enough to get just 5 decimal place accuracy. Later, I learned about the Trapezoid Rule and Simpson's Approximation and I rederived the formula for π using those techniques. With the Trapezoid Rule it took a smaller value of *n* to get 5 decimal place accuracy and my calculator was able to do it in a few hours instead of days. Simpon's Rule took only about 10 minutes to achieve the same level of accuracy.

Class Discussion Problems:

1. Why are some variables pulled out of the summation and limits in the derivation above and others left in?
2. What would the formula be if we used the Trapezoid Rule to derive the area instead of a Reimann Sum?
3. What would the formula be if we used Simpon's Rule to derive the area instead of a Reimann Sum?
4. What value of *n* was needed to get 5 decimal place accuracy for each of the rules above?
5. How can we program a TI-83/84 calculator to calculate π using these rules and the sequence and sum features that are built into the calculator?
6. If you do the above problem, you will notice that your calculator will let you have at most $n = 999$. Any larger and you get an error. What level of accuracy, does your calculator give you with each of the above formulas?
7. Do you see why Simpon's Rule is important?
8. Could you come up with a rule that gets accuracy even faster than Simpon's Rule?